Stable Marriage Problem

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1 Introduction

The Stable Marriage Problem is the problem of finding stable matching between two finite sets of men and women. We call a matching unstable if under it there are a man and a woman who are not married to each other but prefer each other to their actual mates. Each element of the set has a preference list for the elements of the opposite sex. David Gale and Lloyd Shapley proved that, for any equal number of men and women, it is always possible to solve the Stable Marriage Problem and make all marriages stable. They presented an algorithm to do so.

2 Notations and Terminology

- **Matching** : In Mathematical terms, a matching can be defined as a bijection from the set of men M to set of women W. The situation can be generalized as a complete bipartite graph. Each edge represents a marriage between a man and a woman.
- **Stable Matching** : A matching in which there does not exist a man and a woman who prefer each other over their present partners.
- **Unstable Matching** : A matching in which there exist atleast one pair of a man and a woman who prefer each other over their present partners.
- In the following, we will generally denote one of the pair with a capital letter and its counterpart with a small letter, for example, man denoted by A and women denoted by a and so on.
- Also we will use the notation aAb meaning A prefers a to b and similarly for AaB. We also denote a couple or a pair formed as Aa.
- $a \in A$ implies that a is on the preference list of A and similarly for $A \in a$.
- **Incomplete lists** : It is not necessary that the preference lists be complete, so there are cases where they are incomplete(meaning not all the opposite element may not be present in the preference list of an element).

3 Stable Matching

As said in the beginning, a matching between two finite sets(one set is $A, B, ..$ and the other $a, b, ...$) is called a stable matching if there doesnt exist Ab and Ba such that AaB and aAb.

For a more detailed definition of Stable Matching: $(A_1a_1, A_2a_2, ...)$ is a stable matching iff-

- $a_k \in A_k$ and $A_k \in a_k$ for $1 \leq k \leq n$
- there does not exist j and k such that $A_j a_k A_k$ and $a_k A_j a_j$.

To understand more of this,let's go through a example: Here is a matching problem for men and women with their preference lists given:

Men's Choice				
$A:$	c	b	d	a
$B:$	b	a	c	d
$C:$	b	d	a	c
$D:$	c	a	d	b

Women's Choice				
$a:$	A	B	D	C
$b:$	C	A	D	B
$c:$	C	C	C	A
$d:$	B	A	C	D

Let's consider a matching (Aa, Bb, Cc, Dd), it is unstable since A and b prefer each other than their respective partners. So the considered matching is an unstable matching. Next,let's consider (Ad, Ba, Cb, Dc), and we can check that this is a stable matching.

3.1 Incomplete lists

The above example had preference lists which were complete, but there maybe cases where a man(say) did not have all the women in his preference list.These are generally categorised under Incomplete lists. To solve these, there is a method where these lists are converted into Complete lists and then solve them. To convert them into complete lists, we add an imaginary man V, the widower and an imaginary women v, the widow.V will be v's last choice and v will be last on list of V. The women who have incomplete lists will have v added at their end of the list and then the remaining men who were not in the list will be added randomly. Same goes for the men also. Men and women who have complete lists have v and V added as their last choices. This makes the incomplete lists as complete ones.

Theorem 1: *There exists a stable matching for the complete system such that V is married to v if and only if there exists a stable matching for the incomplete system.*

PROOF. This theorem is of the form $p \iff q$. So, we will first prove $p \Rightarrow q$ and then $q \Rightarrow p$.

- For the $p \Rightarrow q$ part.
  Let M = $(A_1, A_2, A_3, ..., A_n, V)$
  and W = $(a_1, a_2, a_3, ..., a_n, v)$
  There exists a stable matching such that v is married to V and a_i is married to A_i
  We know, V is the last preference of v
  $\Rightarrow$ A$_i vV$ $\forall$i $s.t.$ $1 \leq i \leq n$
  For the matching to be stable $a_i A_i V$ holds $\forall is.t.$ $1 \leq i \leq n$
  $\Rightarrow$ a$_i$ comes before V on preference list of A_i $\forall$i $s.t.$ $1 \leq i \leq n$
  $\Rightarrow$ a$_i$ is part of the incomplete list of A_i $\forall$i $s.t.$ $1 \leq i \leq n$
  $\Rightarrow$a$_i A_i$ forms a stable matching where it is independent of V and v.
  Hence, there exists a stable matching for the incomplete list.

- For the $q \Rightarrow p$ part.
  There exists a stable matching for the incomplete system such that A_i is married to a_i and a_i $\in$A$_i$ and A_i $\in$a$_i$.
  Now, we form the complete list for A_i, by adding v after the incomplete list and then place all the remaining a_j's not in his list in random order.
  Here, we already have a stable matching from the incomplete list ,i.e, every A_i has a partner a_i which is ahead of v in the preference list of A_i. Same goes with a_i.This happens $\forall i, s.t$,1$\leq$ i $\leq$ n.So, V has got only v to pair up with and, hence, the complete list has a stable matching with V paired up with v.

Till now, we have assumed that two sets of n elements have a stable matching solution. Infact, it has been proved that there exists a solution and to find it, an algorithm was designed by Gale and Shapley. It is discusssed in detail in the following topics.

4 The Fundamental Algorithm

In the fundamenatal algorithm the men in turn, one by one play the of suitors, making advances to the women, who accept or refuse according to their preference.

4.1 Algorithm

The notations used in the algorithm:

- n : number of men = number of women
- k : no. of couples engaged
- X : the men who approaches the women
- x : women towards whom X makes advances
- Ω : $veryundesirableimaginaryman$

Algorithm 1 Gale-Shapely Algorithm

```
k ← 0
for i < n do
    Marry woman i to Ω;
while k > n do
    Set X to man (k + 1);
    while X ≠ Ω do
        x ← woman on the top of X's list;
        if x prefers X to her present partner then
            temp ← x's present partner;
            Match X and x;
            X ← temp;
            end
        if X ≠ Ω then remove x from X's list;
        end
    k ← k + 1;
    end
```

4.2 Proof of the algorithm

The following are some observations for the proof-

- **Point 1 :** If any woman x removed from the X's list, no stable matching can contain Xx.
  PROOF: Suppose after r proposals the operation remove x from X's list was called with $x = a$ and $X = A$,then one of two cases is possible

  1. Man A made advances to a, but she prefers her current partner B.
  2. Woman a was engaged to A but she left him after receiving better proposal from B.

  So in both of the cases a prefers B to A.
  For the sake of contradiction let us assume that a stable matching exists in which A is married to a.
  Now we apply induction on the number of proposals.
  Since BaA, B must be married to someone occurring before a in his list of preferences. Clearly during the algorithm the case would be that either a was B's first choice, or B was rejected by some other woman say b. This would have surely occurred within $r - 1$ proposals.

  If a was first in B's list of preferences, then B cannot marry anyone better than a, so we arrive at contradiction.

  If a was not B's first choice, then he must have been rejected by the women to whom he is married in the stable matching. Lets say that the woman is b. b would have rejected him in at most $r - 1$ proposals. Now all of the above procedure repeats with A replaced by B, a replaced by b and r replaced by $r - 1$.

The above algorithm can only terminate at a contradiction. It will surely terminate because at maximum it can go upto r iterations. This is because number of proposals decreases by 1 at each step but need to be greater than zero. Hence Contradiction.

So a Stable Matching containing Aa cannot exist.

- **Point 2 :** If X prefers x to his fiancée and stable matching does not contain Xx, it means x has rejected him for another.

- **Point 3 :** Two women cannot be the fiancée of a man.

- **Point 4 :** A woman's situation never worsens throughout the course of the algorithm.
PROOF : Initially all the women are married to Ω and they marry to any other men on proposal only when he is above the one they are engaged in their preference list.

- **Point 5 :** The preference list of each man never becomes empty.
PROOF : Let's assume that he(say X) ended up having no partner and every women has a partner. But by point 3, there should be n partners and by point 4, we can say that the women do not have Ω as their partner. Besides X, there are only $n-1$ men which is a contradiction. Therefore, X will have a partner.

- **Point 6 :** Once women is matched shee never becomes unmatched , she only "trades up".

- **Point 7 :** Man-optimal assignment.Each man receives best valid partner(according to his preferences).

- **Point 8 :** The matching obtained is stable.
PROOF : Suppose Xx is a pair from the algorithm, and X prefers some y to x. By point 2 and point 4, we can say that y prefers some other man over X. Hence, this algorithm gives a stable matching solution.

- **Point 9 :** The algorithm terminates.
PROOF : Once a woman becomes attached, she remains engaged, but can change a partner for a better mate that proposes to her. That makes this algorithm a greedy algorithm for the women(Point 4). A man will eliminate a choice from his list during each iteration, thus if the rounds continue long enough, he will get rid of his entire preference list entries and there will be no one left to propose too. Contradiction(Point 5). Therefore all women and men are married and the algorithm terminates.

5 Existence of multiple stable matching and Conflict of Interests

One may argue that a system can have more than one stable matching. Which is indeed true, rather, it is a common phenomenon. Some systems may even have as many as $3^{k/3}$ stable matchings (where k is the largest integer divisible by 3 less than or equal to the number of men/women). One such example is demonstrated below.

Men's Choice			
A :	a	b	c
B :	b	c	a
C :	c	a	b

Women's Choice			
a :	C	B	A
b :	A	C	B
c :	B	A	C

It can be easily verified that three stable matchings are possible, which are as follows:

$$\begin{array}{ccc} Aa & Bb & Cc \\ Ab & Bc & Ca \\ Ac & Ba & Cb \end{array}$$

One particular thing which must be noted is that conflicts may arise. A stable matching which matches all men to their first choice, matches all women to their last choice or vice-verse. A matching which is best for men needs not be the best for women too. However, that particular matching would be the worst for women. i.e., in any other stable matching, no woman will be matched to a man whom she prefers less than her partner in the best matching for men. The optimality of the matching is dependent on the algorithm used to derive the matching. The fact that "the best for the men is the worst for the women" is a special case of a more general result.

Theorem 2: If one stable matching contains the couple Aa, and another contains couples Ab and Ba, then either

$$bAa \text{ and } AaB,$$

or

$$aAb \text{ and } BaA.$$

PROOF: In other words, the theorem says that the situation can't get better/worse for both A and a simultaneously. If it gets better for one, then it would worse for other. This can be proved by method of contradiction. It is quite clear that in second matching, the situation of A and a can't be worse than that in first. If so, this would make the second matching unstable. Therefore, it remains to show that the situation can't improve for both at the same time.

Let $A = X_0$, $B = X_r$ $(r > 0)$, $a = x_0$, $b = x_1$. Also, it is assumed that situations get better for both x_0 and X_0, i.e.:

$$x_1 X_0 x_0 \text{ and } X_r x_0 X_0.$$

In the first matching, X_0 is not matched to x_1, it implies that x_1 must have been matched to some X_1, whom she prefers above X_0. i.e. $X_1 x_1 X_0$. Similarly, in second matching, x_1 is not matched to X_1, so X_1 must have been matched to some x_2 whom he prefers over x_1. i.e. $x_2 X_1 x_1$. This can be carried forward for x_2, X_2, x_3,..... The two Stable matchings can be represented as follows:

$$X_0x_0, \quad X_1x_1, \quad X_2x_2.... \text{ First stable matching}$$

$$X_0x_1, \quad X_1x_2, \quad X_2x_3.... \text{ Second stable matching}$$

Also $x_{k+1} X_k x_k$ and $X_{k+1} x_{k+1} X_k$ for all $k \geq 0$. But k is finite and must be less than n, i.e for X_r, $r \leq n-1$. If $(r = n-1)$, it is already proved $X_{n-1} x_{n-1} X_{n-2}$. i.e. for second matching to be stable, $x_0 X_{n-1} x_{n-1}$. But this makes the first matching unstable. ($X_{n-1} x_0$ must be present in First match also). for $r < n-1$, $X_r x_r X_{r-1}$. But since, second matching is stable and X_r is matched to x_0, then it must be $x_0 X_r x_r$. This thing again makes the first matching unstable. i.e. in both the cases, The assumptions get contradicted. i.e. Situation can't get better for both men and women simultaneously.
Thus, Theorem 2 is proved.

Theorem 3 : If there exists a stable matching with V married to v in complete system, then, for all stable matchings of this system V is married to v.
PROOF : Let us assume that there exists a stable matching such that V and v are not married with each other, they are engaged with any other woman a and man A, so since V is the last preference of women v and vice versa. So, when V is married to a and v is married to A the matching improves for both v V simultaneously which is not possible as we proved in Theorem 2. Hence Contradiction.
Therefore, V is not married to v in any of the stable matchings.

6 Calculations of Mean number of proposals

6.1 Principle of Deferred Decisions

Game of Clock Solitaire In this game we divide the deck in 13 stacks of 4 cards each and place them in a circle to mimic the hands of a clock with last pile at the centre of the clock. We turn the top card from centre pile and place it face up under the pile of that card's number i.e. place 7 under 7th pile, J under 11th pile, Ace under 1st pile, Q under 12th, K in the middle pile, etc. Now we turn the top card from the pile under which we just placed a card and again place it face up under its correct pile. The game is won if we can turn all the cards in this fashion.

We lose the game if the fourth king is turned before all the other cards are turned. This is because before turning 4th king three kings have already been picked so from the central deck 3 cards are turned after turning first card, so it is empty and we cannot turn any card from it.

We analyse this using the principle of deferred decisions. In this method the choice of value is not made before the moment it is turned over.
We turn over the cards by the rules of the game until 4th king is drawn and then pick the remaining cards by any arbitrary rule till all the cards are drawn. The number of permutations thus obtained is equal to the number of permutations of a deck of 52 cards. Each of the permutation obtained above is equally likely in both the cases. So the probability of winning is equal to probability that last card is king in a well shuffled deck i.e. 1/13.

Principle of deferred decisions: This principle uses an idea that random choices are not made all in advance but the algorithm makes random choices as it needs them. It is like the general principle of laziness that don't do something today that you can do tomorrow.

6.2 Counting: Mean number of proposals and Amnesia

In this case we assume that all men are amnesiac. None of the men has a preference list neither do they remember which women they had already proposed, so they randomly proposes any woman. The Actual mean would be the number of mean proposals in this situation minus the redundant proposals (all the cases when a man proposes a woman whom he had already proposed). In calculating actual mean, these cases would not be considered. We do not calculate the actual mean. We calculate mean in this case, which serves as a upper bound for our required quantity.

A random sequence of women is generated, which may consists any women any number of times, but the sequence ends the moment all the women have appeared in the list at least once. Men proceed by proposing to women according to the generated sequence one by one. i.e. the first man proposes the first woman in the sequence(and is obviously accepted), the second man then proposes to second woman in the sequence. The rule is that the next man won't make any proposals unless the previous men have been engaged(Any rejected/deserted man must also be engaged by proposing to next women in sequence). The sequence and the ritual of making proposal ends together. And thus, a stable matching is obtained. Note that, this matching might not be the best matching for women, also, there is no meaning of stable matching/marriage for any man.

The number of proposals made in this procedure is equal to the number of elements in the sequence. Now, the problem is calculate the expected number of elements in it. This problem is exactly similar to the coupon collection problem.

6.2.1 Coupon Collection Problem

Suppose, a detergent company provides free stickers with the detergent packets. There are let say n type of stickers. The aim is to collect all n stickers. Each packet has only one sticker and we can't know which sticker comes with the packet before buying it. The problem is to calculate the expected number of detergent packets we'll have to buy to collect them all.

If we arrange the stickers in the order we found them(the sticker we found first, leads the sequence and the last sticker is on the last place), the generated sequence is exactly the sequence we generated in our problem. The stickers can be considered as women and buying detergent packets can be compared to proposing a woman(getting rejected/deserted would be similar to already having the coupon). So, to calculate the upper bound for mean number of proposals(in case of n men/women), we calculate the expected number of packets we have to buy to collect n different stickers

Let there be n distinct coupons, and each time one buys a box he gets a random coupon. The problem is to find the number of boxes to be bought on average to obtain all coupons.

Let p_k be the probability that exactly k boxes are necessary. Let q_k be the probability that at least k boxes are necessary,then

$$q_1 = p_1 + p_2 + p_3 + ...$$
$$q_2 = p_2 + p_3 + ...$$

$$q_k = p_k + p_{k+1} + ...$$

The mean number of boxes used is

$$p_1 + 2p_2 + 3p_3 + ... = q_1 + q_2 + ...$$

Here $q_i = (\frac{m}{n})^{i-1}$ is the probability of finding a new coupon in or after i^{th} box.
So,

$$q_1 + q_2 + ... = 1 + \frac{m}{n} + \left(\frac{m}{n}\right)^2 + ...$$
$$q_1 + q_2 + q_3 + ... = 1 + \frac{m}{n} + \left(\frac{m}{n}\right)^2 + ... = \frac{n}{n-m}$$

The total number of boxes one must buy to get all coupons is thus, let say $E(n)$

$$E(n) = \frac{n}{n-0} + \frac{n}{n-1} + \frac{n}{n-2} + ... + \frac{n}{n-(n-1)} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}\right)$$
$$= nH_n$$

where H_n is the harmonic sum of the first n natural numbers

6.3 Partial Amnesia

We now assume that men are partially amnesiac, i.e. All they remember is the last woman they proposed. So, this problem is just like the coupon collection problem with the constraint that two consecutive coupons aren't same. Now, given this, the values of $q_i s$ change as follows.

$$q_i = 1; \qquad q_2 = \frac{m}{n}; \qquad q_3 = \frac{m}{n}\frac{m-1}{n-1}; \qquad q_4 = \frac{m}{n}\left(\frac{m-1}{n-1}\right)^2;$$

$$q_1 + q_2 + q_3 + ... = 1 + \frac{m}{n}\left(1 + \frac{m-1}{n-1} + \left(\frac{m-1}{n-1}\right)^2 + ...\right)$$
$$= \frac{n}{n-m} - \frac{m}{n(n-m)}$$

It is clearly evident that $q_1 = 1$. Now, to take atleast two proposals, the probability that the first proposal to one of which we already had is m/n. Now, thereafter, for $i = 3$ and second coupon, the probability that coupon is among the m coupons we already have, given it is not similar the last coupon is $(m-1)/(n-1)$. The mean $E(n)$ can be calculated as:

$$E(n) = \sum_{m=0}^{n-1} \frac{n^2 - m}{n(n-m)} = \sum_{i=1}^{n} \frac{n^2 - (n-m)}{nm}$$
$$= (n-1)\left(\sum_{m=1}^{n} \frac{1}{m}\right) + \frac{1}{n}\left(\sum_{m=1}^{n} 1\right)$$
$$= (n-1)H_n + 1.$$

Theorem 4 : For every preference matrix of the women ,the mean number of proposal made in the course of the algorithm ending in an optimal solution for the men is at most $(n-1)H_n + 1$.

7 Applications

Stable Marriage Problem used in Economics, Computer Science and Mathematics for finding the stable matching between two equally sized sets of elements given an ordering of preferences for each element. Gale-Shapley Algorithm finds various application in real life situations like allocations of students to different universities based on the merit list. This algorithm has been used for allocation of seats in 2015 by JoSAA.

8 References

Following books/articles has been referred:

- Knuth, D. E. (1976). Stable Marriage and Its Relation to Other Combinatorial Problems. [translated by Martin Goldstein]
- Wikipedia page on Stable Marriage Problem:
  `https://en.wikipedia.org/wiki/stable marriage problem.`